LUCIE: An Evaluation and Selection Method for Stochastic Problems – Appendix

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1 NOTATIONS

We recall here the definition of the different sets we used in the paper and introduce some more for the purpose of the proofs.

Consider generation g, while applying LUCIE with parameters μ, λ, ϵ, δ. All the sets described below are illustrated in Figure A.1 and correspond to the same notations as [2]. We write $\textsc{Top}^g$ the set of the μ individuals with highest expected fitness, and, $\textsc{Bot}^g$ the λ remaining individuals: For $\epsilon \in [0, 1]$, we define $\textsc{Good}^g$, the set of (ϵ, μ)-optimal individual, and $\textsc{Bad}^g$, as

$$\textsc{Good}^g \overset{\text{def}}{=} \left\{ i \in \textsc{Ind}^g,\ f_i \geq \min_{j \in \textsc{Top}^g} f_j - \epsilon \right\}$$

$$\textsc{Bad}^g \overset{\text{def}}{=} \textsc{Ind}^g \setminus \textsc{Good}^g$$

The objective of elitism is to be able to return μ individuals in $\textsc{Good}^g$. Obviously, the algorithm is unaware of what individuals are in $\textsc{Good}^g$ or $\textsc{Bad}^g$. Instead, it maintains the sets $\textsc{High}^{g,t}$ and $\textsc{Low}^{g,t}$, respectively containing the *believed* best μ individual and the remainder. This ordering is performed based on the empirical fitness. Further, we define c as the mean between the fitness of the "worst" individual in $\textsc{Top}^g$ and the fitness of the "best" individual in $\textsc{Bot}^g$:

$$c \overset{\text{def}}{=} \frac{1}{2} \left(\min_{i \in \textsc{Top}^g} f_i + \max_{i \in \textsc{Bot}^g} f_i \right). \tag{A.1}$$

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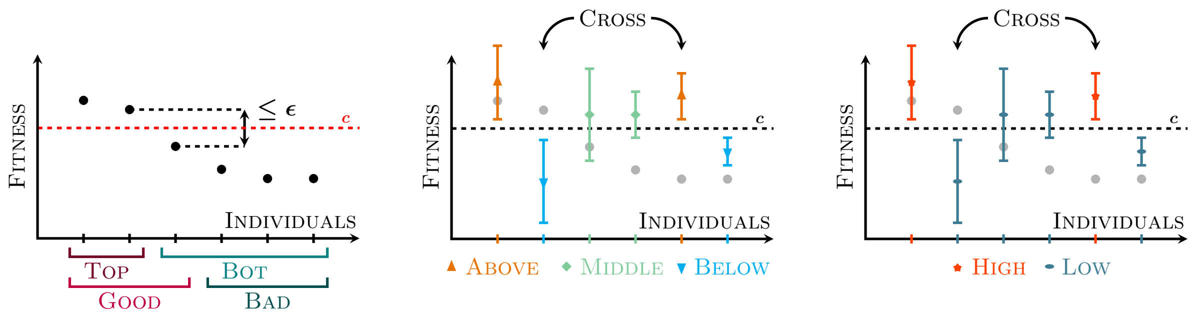

Fig. A.1. Illustration of the different sets for $\mu = 2, \lambda = 4$. The circle markers represent true fitness of individuals while the triangles, diamonds, stars and ellipses represent empirical fitness means. Additionally, those empirical means are surrounded by confidence intervals.

Based on c, we can define the sets $\textsc{Above}^{g,t}$, $\textsc{Below}^{g,t}$, and, $\textsc{Middle}^{g,t}$ as follows:

$$\begin{aligned} \textsc{Above}^{g,t} &\overset{\text{def}}{=} \left\{ i \in \textsc{Ind}^g, \hat{f}_i^{g,t} - \beta\left(u_i^{g,t}, t\right) \geq c \right\}, \\ \textsc{Below}^{g,t} &\overset{\text{def}}{=} \left\{ i \in \textsc{Ind}^g, \hat{f}_i^{g,t} + \beta\left(u_i^{g,t}, t\right) \leq c \right\}, \\ \textsc{Middle}^{g,t} &\overset{\text{def}}{=} \textsc{Ind}^g \setminus \left(\textsc{Above}^{g,t} \cup \textsc{Below}^{g,t}\right). \end{aligned} \tag{A.2}$$

As c separates the fitness of individuals in $\textsc{Top}^g$ from the fitness of individuals in $\textsc{Bot}^g$, we wish that individuals in $\textsc{Above}^{g,t}$ (respectively in $\textsc{Below}^{g,t}$) and $\textsc{Top}^g$ (respectively in $\textsc{Bot}^g t$) are the same. We call $\textsc{Cross}_i^{g,t}$ the event that this is not the case for individual $i \in \textsc{Ind}^g$, and $\textsc{Cross}^{g,t}$ the event that $\textsc{Cross}_i^{g,t}$ is true for at least one individual. Formally,

$$\textsc{Cross}_i^{g,t} \overset{\text{def}}{=} \begin{cases} i \in \textsc{Below}^{g,t} & \text{if } i \in \textsc{Top}^g, \\ i \in \textsc{Above}^{g,t} & \text{if } i \in \textsc{Bot}^g. \end{cases}$$

$$\textsc{Cross}^{g,t} \overset{\text{def}}{=} \exists i \in \textsc{Ind}^g, \textsc{Cross}_i^{g,t}.$$

Additionally, we define the event that an individual is "needy", meaning that it belongs to the $\textsc{Middle}^{g,t}$ set, and that its confidence bound is larger than $\epsilon/2$:

$$\textsc{Needy}_i^{g,t} \overset{\text{def}}{=} \left(i \in \textsc{Middle}^{g,t}\right) \wedge \left(\beta\left(u_i^{g,t}, t\right) > \frac{\epsilon}{2}\right).$$

Finally, we define the event of termination at step t as true when the termination criterion of Equation 2 has been met during step t or before:

$$\textsc{Term}^{g,t} \overset{\text{def}}{=} \exists t' \leq t, \hat{f}_{l^{t'}}^{g,t'} + \beta\left(u_{l^{t'}}^{g,t'}, t'\right) < \hat{f}_{h^{t'}}^{g,t'} - \beta\left(u_{h^{t'}}^{g,t'}, t'\right) + \epsilon,$$

where the sampling candidates h^t and l^t are defined according to Equation 1.

2 PROOF OF THEOREM 4.1

Claim: Let g be the current generation and $\beta : \mathbb{N}^2 \to \mathbb{R}^+$ a function such that

$$\sum_{i=1}^{n} \sum_{t=1}^{\infty} \sum_{u=1}^{\infty} \exp\left(-2u\beta\left(u, t\right)^2\right) \leq \delta,$$

if LUCIE terminates (Equation 2), the probability of returning a non-(ϵ, μ)-optimal individual is at most δ, with $0 < \delta \leq 0.5$.

Proof. The proof of this result differs from the one of Theorem 1 of [2] in that the number of samples of the fitness of an individual can be larger than the number of steps of the current generations. Indeed, as samples are kept between generations, this number can get arbitrarily large. Let us write F^g the event of failing, *i.e.*, reaching the termination criterion of Equation 2 and returning a non-(ϵ, μ)-optimal individual, during generation g. F_t^g the event of failing during generation g at step t. We are interested in bounding $\mathbb{P}(F^g)$. Applying the union bound, we have that

$$\mathbb{P}(F^g) = \mathbb{P}\left(\bigcup_{t=1}^{\infty} F_t^g\right) \leq \sum_{t=1}^{\infty} \mathbb{P}\left(F_t^g\right). \tag{A.3}$$

Let us upper-bound the probability of F_t^g. At step t, we define the event that an individual $i \in \textsc{Ind}^g$ is *well-behaved* if its empirical fitness is larger (respectively lesser) than its true fitness minus (respectively plus) $\beta(u, t)$ if $i \in \textsc{Top}^g$ (respectively $i \in \textsc{Bot}^g$):

$$WB_i^{g,t}(u) \overset{\text{def}}{=} \begin{cases} \hat{f}_i^{g,t} \geq f_i - \beta(u,t) & \text{if } i \in \textsc{Top}^g \text{, and,} \\ \hat{f}_i^{g,t} \leq f_i + \beta(u,t) & \text{if } i \in \textsc{Bot}^g. \end{cases} \tag{A.4}$$

Intuitively, a well-behaved individual is one whose estimated empirical fitness would yield a correct guess of belonging to $\textsc{Top}^g$ or $\textsc{Bot}^g$. Let us show that the event F_t^g implies the event $\left(\exists i \in \textsc{Ind}^g, \neg WB_i^{g,t}(u_i^{g,t})\right)$. In plain words, if LUCIE fails at step t of generation g, then, there necessarily exists a non well-behaved individual. Suppose that F_t^g is true. This means an individual $i \in \textsc{Bad}^g$ has been recommended, *i.e.*, $i \in \textsc{High}^{g,t}$. Necessarily, an (ϵ, μ)-optimal individual $j \in \textsc{Good}^g$ has not been recommended, *i.e.*, $j \in \textsc{Low}^{g,t}$. Now, let us suppose that i is well-behaved, *i.e.*,

$$\hat{f}_i^{g,t} \leq f_i + \beta\left(u_i^{g,t}, t\right).$$

Since $i \in \textsc{High}^{g,t}$ and $j \in \textsc{Low}^{g,t}$, *and* the stopping criterion (Equation 2) has been met, we have that

$$\hat{f}_j^{g,t} + \beta\left(u_j^{g,t}, t\right) \leq \hat{f}_i^{g,t} - \beta\left(u_i^{g,t}, t\right) + \epsilon.$$

Since $i \in \textsc{Bad}^g$ and $j \in \textsc{Good}^g$, we have that

$$f_i < f_j - \epsilon.$$

Combining the three previous inequalities, we get that

$$\begin{aligned} f_j &> f_i + \epsilon \\ &> \hat{f}_i^{g,t} - \beta\left(u_i^{g,t}, t\right) + \epsilon \\ &> \hat{f}_j^{g,t} + \beta\left(u_j^{g,t}, t\right), \end{aligned}$$

which implies that j is not well-behaved. As a conclusion, either i or j is not well-behaved, thus:

$$F_t^g \implies \left(\exists i \in \textsc{Ind}^g, \neg WB_i^{g,t}(u_i^{g,t})\right). \tag{A.5}$$

Note that the random variable $u_i^{g,t}$ is greater than 1, regardless of the sampling strategy $\mathcal{S}$, as all individuals must be sampled at least once. We inject the previous result in Equation A.3:

$$
\begin{aligned}
\mathbb{P}\left(F^g\right) &\leq \sum_{t=1}^{\infty} \mathbb{P}\left(F_t^g\right) && \text{(from A.3)}\\
&\leq \sum_{t=1}^{\infty} \mathbb{P}\left(\exists i \in \textsc{Ind}^g, \neg WB_i^{g,t}(u_i^{g,t})\right) && \text{(from A.5)}\\
&\leq \sum_{t=1}^{\infty} \sum_{j=1}^{n} \mathbb{P}\left(\neg WB_j^{g,t}(u_j^{g,t})\right) && \text{(union bound on } j\text{)}\\
&\leq \sum_{t=1}^{\infty} \sum_{j=1}^{n} \sum_{u=1}^{\infty} \mathbb{P}\left(\neg WB_j^{g,t}(u)\right) && \text{(union bound on } u\text{)}\\
&\leq \sum_{t=1}^{\infty} \sum_{j=1}^{n} \sum_{u=1}^{\infty} \exp\left(-2u\beta\left(u,t\right)^2\right). && \text{(Hoeffding's bound)}
\end{aligned}
$$

More precisely, about the use of the Hoeffding's bound in the last step, the probability of an individual not being well-behaved is the probability that its empirical fitness deviates from its true fitness by at least $\beta(u,t)$. By symmetry between the two cases where the individual belongs to $\textsc{Top}^g$ or $\textsc{Bot}^g$, the same one-sided Hoeffding's bound applies. Finally, choosing β such that

$$\sum_{i=1}^{n} \sum_{t=1}^{\infty} \sum_{u=1}^{\infty} \exp\left(-2u\beta\left(u,t\right)^2\right) \leq \delta,$$

implies $\mathbb{P}\left(F^g\right) \leq \delta$, which concludes the proof. □

3 PROOF OF THEOREM 4.2

Claim: At generation g, the expected sample complexity of LUCIE is $\mathcal{O}\left(\left(H^{g,\frac{\epsilon}{2}} \ln\left(\frac{H^{g,\frac{\epsilon}{2}}}{\delta}\right)\right)^{\frac{1}{\gamma}}\right)$, with γ a constant value such that $0 < \gamma < 0.57$.

The proof of this result shares a similar reasoning as the one of Theorem 6 of [2]. The main differences are brought by the fact that we here assume that previous bandit problems have been solved during former generations and thus, at the initialization of the new problem (*i.e.*, current generation), some arms (or individuals) may already have been sampled.

We first introduce six Lemmas. To ease the comprehension, we encourage the reader to directly skip to the proof of the main theorem (page 14), which points back to the intermediate lemmas.

Lemma 3.1. *Consider step t of generation g. By applying LUCIE with β defined in Equation 4,*

$$\mathbb{P}\left(\textsc{Cross}^{g,t}\right) \leq \frac{\delta\zeta(2)}{kt^4},$$

where ζ is the Riemann zeta function.

Proof. Without loss of generality, let i be an individual in $\textsc{Top}^g$. By definition, at step t, with c defined as the mean between the fitness of the "worst" individual in $\textsc{Top}^g$ and the fitness of the "best" individual in $\textsc{Bot}^g$

(Equation A.1),

$$
\begin{aligned}
\mathbb{P}\left(\textsc{Cross}_i^{g,t}\right) &= \mathbb{P}\left(\hat{f}_i^{g,t} + \beta\left(u_i^{g,t}, t\right) \leq c\right) \\
&= \mathbb{P}\left(\hat{f}_i^{g,t} \leq f_i - \left(f_i + \beta\left(u_i^{g,t}, t\right) - c\right)\right) \\
&\leq \sum_{u=1}^{\infty} \mathbb{P}\left(\hat{f}_i^{g,t} \leq f_i - \left(f_i + \beta\left(u, t\right) - c\right)\right) && \text{(union bound on } u\text{)} \\
&\leq \sum_{u=1}^{\infty} \exp\left(-2u\left(f_i + \beta\left(u, t\right) - c\right)^2\right) && \text{(Hoeffding's bound)} \\
&\leq \sum_{u=1}^{\infty} \exp\left(-2u\left(\beta\left(u, t\right)\right)^2\right) && \text{(as } f_i - c \geq 0\text{)} \\
&\leq \sum_{u=1}^{\infty} \frac{\delta}{nkt^4u^2} \\
&\leq \frac{\delta\zeta(2)}{nkt^4}
\end{aligned}
$$

The same applies for $i \in \textsc{Bot}^g$. The remainder of the proof follows naturally by applying the union bound over all individuals.

$$
\mathbb{P}\left(\textsc{Cross}^{g,t}\right) = \mathbb{P}\left(\bigcup_{i=1}^{n} \textsc{Cross}_i^{g,t}\right) \leq \sum_{i=1}^{n} \mathbb{P}\left(\textsc{Cross}_i^{g,t}\right) \leq \sum_{i=1}^{n} \frac{\delta\zeta(2)}{nkt^4} \leq \frac{\delta\zeta(2)}{kt^4}
$$

□

Lemma 3.2. *Consider step t of generation g. We define the number of samples $u_i^*(t) \in \mathbb{N}$ as*

$$
u_i^*(t) \overset{def}{=} \left\lceil \left(\frac{2}{\left[\Delta_i \wedge \frac{\epsilon}{2}\right]^2} \ln\left(\frac{nkt^4}{\delta}\right) \right)^{\frac{1}{\gamma}} \right\rceil,
$$

with γ any constant value verifying $0 < \gamma < 0.57$. Then we have that $\forall u \geq u_i^*(t), \beta(u,t) \leq \frac{1}{2}\left[\Delta_i \wedge \frac{\epsilon}{2}\right]$.

Proof. The key argument of this proof consists in upper-bounding β by a looser bound $\bar{\beta}$ that we define as

$$
\bar{\beta}\left(u, t\right) \overset{\text{def}}{=} \sqrt{\frac{1}{2u^\gamma} \ln\left(\frac{nkt^4}{\delta}\right)}.
$$

We first prove that β is indeed upper-bounded by $\bar{\beta}$ for any u, then we prove that when $u \geq u_i^*(t)$, $\bar{\beta}$ is upper-bounded by $\frac{1}{2}\left[\Delta_i \wedge \frac{\epsilon}{2}\right]$.

Upper-bound of β by $\bar{\beta}$. Consider $u, t \in \mathbb{N}$,

$$
\begin{aligned}
\beta(u,t) \leq \bar{\beta}(u,t) &\iff \sqrt{\frac{1}{2u}\ln\left(\frac{nkt^4u^2}{\delta}\right)} \leq \sqrt{\frac{1}{2u^\gamma}\ln\left(\frac{nkt^4}{\delta}\right)} \\
&\iff \frac{\ln\left(\frac{nkt^4u^2}{\delta}\right)}{\ln\left(\frac{nkt^4}{\delta}\right)} \leq u^{1-\gamma} \\
&\iff \ln\left(\frac{\ln\left(\frac{nkt^4u^2}{\delta}\right)}{\ln\left(\frac{nkt^4}{\delta}\right)}\right) \leq (1-\gamma)\ln(u) \\
&\iff \gamma \leq 1 - \frac{1}{\ln(u)}\ln\left(\frac{\ln\left(\frac{nkt^4u^2}{\delta}\right)}{\ln\left(\frac{nkt^4}{\delta}\right)}\right) \\
&\iff \gamma \leq 1 - \frac{1}{\ln(u)}\ln\left(1 + \frac{2\ln(u)}{\ln\left(\frac{nkt^4}{\delta}\right)}\right) \qquad \text{(A.6)}
\end{aligned}
$$

We now show that the last statement is true for any value of t, u and for any γ such that $0 < \gamma < 0.57$. First, remark that, as $x \mapsto \ln(1+x)$ is bounded by the identity function on $\mathbb{R}^+$, the right hand side of the previous inequality is lower-bounded by

$$
1 - \frac{2}{\ln\left(\frac{nkt^4}{\delta}\right)}.
$$

This quantity is itself lower-bounded by its minimum value, reached with the minimum values of $n = 2$, $t = 2$, and the maximum value of $\delta = 0.5$. Indeed, the population size cannot be strictly less than 2 by assumption, the number of steps t is at least equal to this number as all individuals are sampled at the beginning of a generation. By replacing the values, we get

$$
1 - \frac{2}{\ln\left(\frac{nkt^4}{\delta}\right)} \geq 1 - \frac{2}{\ln\left(\frac{2\times\pi^6\times2^4}{0.5\times540}\right)} \approx 0.577.
$$

Overall, as $\gamma < 0.57$, we have for any value of t and u that

$$
\gamma < 1 - \frac{2}{\ln\left(\frac{nkt^4}{\delta}\right)} \leq 1 - \frac{1}{\ln(u)}\ln\left(1 + \frac{2\ln(u)}{\ln\left(\frac{nkt^4}{\delta}\right)}\right),
$$

which validates the statement made in Equation A.6. This statement is equivalent to $\beta(u,t) \leq \bar{\beta}(u,t)$ for any values of u, t, which validates that $\bar{\beta}$ is an upper-bound on β.

Upper-bound of β by $\frac{1}{2}\left[\Delta_i \wedge \frac{\epsilon}{2}\right]$. The result is straightforward by showing that $\bar{\beta}$ is upper-bounded by $\frac{1}{2}\left[\Delta_i \wedge \frac{\epsilon}{2}\right]$ if $u \geq u_i^*(t)$.

$$\begin{aligned}
\bar{\beta}(u,t) \leq \frac{1}{2}\left[\Delta_i \wedge \frac{\epsilon}{2}\right] &\iff \sqrt{\frac{1}{2u^\gamma}\ln\left(\frac{nkt^4}{\delta}\right)} \leq \frac{1}{2}\left[\Delta_i \wedge \frac{\epsilon}{2}\right] \\
&\iff \frac{1}{u^\gamma}\ln\left(\frac{nkt^4}{\delta}\right) \leq \frac{1}{2}\left[\Delta_i \wedge \frac{\epsilon}{2}\right]^2 \\
&\iff u \geq \left(\frac{2}{\left[\Delta_i \wedge \frac{\epsilon}{2}\right]^2}\ln\left(\frac{nkt^4}{\delta}\right)\right)^{\frac{1}{\gamma}} \\
&\iff u \geq u_i^*(t).
\end{aligned}$$

The bound on β follows immediately with the fact that $\bar{\beta}$ is an upper-bound on β. □

Lemma 3.3. *Consider step t of generation g, for any constant value $C_1 > 3^{\frac{2}{\gamma}}$, we have that*

$$\mathbb{P}\left(\exists i \in \textit{Ind}^g, \left(u_i^{g,t} > C_1 u_i^*(t)\right) \wedge \textit{Needy}_i^{g,t}\right) \leq C_2 \frac{\delta H^{g,\frac{\epsilon}{2}}}{nkt^4},$$

with $C_2 > 0$ another constant value.

Proof. We distinguish between the two cases of the relative position of Δ_i and $\frac{\epsilon}{2}$. We will use the set $\textsc{Middle}^{g,t}$, defined in Equation A.2, corresponding to the individuals whose confidence interval comprises the value c (Equation A.1). If $\Delta_i \leq \frac{\epsilon}{2}$, the result follows easily, consider $i \in \textsc{Ind}^g$:

$$\begin{aligned}
\mathbb{P}\left(\left(u_i^{g,t} > C_1 u_i^*(t)\right) \wedge \textsc{Needy}_i^{g,t} \;\middle|\; \Delta_i \leq \frac{\epsilon}{2}\right) &= \mathbb{P}\left(\left(u_i^{g,t} > C_1 u_i^*(t)\right) \wedge \left(i \in \textsc{Middle}^{g,t}\right) \wedge \left(\beta\left(u_i^{g,t}, t\right) > \frac{\epsilon}{2}\right)\right) \\
&\leq \mathbb{P}\left(\left(u_i^{g,t} > C_1 u_i^*(t)\right) \wedge \left(\beta\left(u_i^{g,t}, t\right) \geq \frac{\epsilon}{2}\right)\right) \\
&\leq \sum_{u=C_1 u_i^*(t)+1}^{\infty} \mathbb{P}\left(\beta(u,t) \geq \frac{\epsilon}{2}\right). \\
&\leq 0
\end{aligned}$$

The last inequality comes from the fact that $C_1 u_i^*(t) > u_i^*(t)$ and we know from Lemma 3.2 that $\beta(u,t) < \left[\Delta_i \wedge \frac{\epsilon}{2}\right] = \frac{\epsilon}{2}$ for any $u \geq u_i^*(t)$. Hence, $\mathbb{P}\left(\beta(u,t) \geq \frac{\epsilon}{2}\right) = 0$, for such a value of u, which proves the first case.

Consider now the less trivial case where $\left[\Delta_i \wedge \frac{\epsilon}{2}\right] = \Delta_i$. Without loss of generality, consider $i \in \textsc{Top}^g$.

$$\begin{aligned}
\mathbb{P}\left(\left(u_i^{g,t} > C_1 u_i^*(t)\right) \wedge \textsc{Needy}_i^{g,t} \,\middle|\, \Delta_i > \frac{\epsilon}{2}\right) &= \mathbb{P}\left(\left(u_i^{g,t} > C_1 u_i^*(t)\right) \wedge \left(i \in \textsc{Middle}^{g,t}\right) \wedge \left(\beta\left(u_i^{g,t}, t\right) > \frac{\epsilon}{2}\right)\right) \\
&\leq \mathbb{P}\left(\left(u_i^{g,t} > C_1 u_i^*(t)\right) \wedge \left(i \in \textsc{Middle}^{g,t}\right)\right) \\
&\leq \mathbb{P}\left(\left(u_i^{g,t} > C_1 u_i^*(t)\right) \wedge \left(\hat{f}_i^{g,t} - \beta\left(u_i^{g,t}, t\right) \leq c\right)\right) \\
&\leq \sum_{u=C_1 u_i^*(t)+1}^{\infty} \mathbb{P}\left(\hat{f}_i^{g,t} - \beta(u,t) \leq c\right) \\
&\leq \sum_{u=C_1 u_i^*(t)+1}^{\infty} \mathbb{P}\left(\hat{f}_i^{g,t} \leq f_i - \left(f_i - c - \beta(u,t)\right)\right) \\
&\leq \sum_{u=C_1 u_i^*(t)+1}^{\infty} \exp\left(-2u\left(f_i - c - \beta(u,t)\right)^2\right) \\
&\leq \sum_{u=C_1 u_i^*(t)+1}^{\infty} \exp\left(-2u\left(f_i - c - \bar{\beta}(u,t)\right)^2\right)
\end{aligned}$$

The last inequality comes from the fact that $\beta(u,t) \leq \bar{\beta}(u,t) \leq \frac{\Delta_i}{2}$ on one hand, and, $\frac{\Delta_i}{2} \leq f_i - c \leq \Delta_i$, on the other hand (can be shown by using the definition of Δ_i and the triangle inequality). This allows writing that

$$\left(f_i - c - \bar{\beta}(u,t)\right)^2 \leq \left(f_i - c - \beta(u,t)\right)^2,$$

hence we can upper-bound the right hand side by replacing β by $\bar{\beta}$. Developing the definition of $\bar{\beta}$, we get the following:

$$\begin{aligned}
\mathbb{P}\left(\left(u_i^{g,t} > C_1 u_i^*(t)\right) \wedge \textsc{Needy}_i^{g,t} \,\middle|\, \Delta_i > \frac{\epsilon}{2}\right) &\leq \sum_{u=C_1 u_i^*(t)+1}^{\infty} \exp\left(-2u\left(f_i - c - \sqrt{\frac{1}{2u^\gamma}\ln\left(\frac{nkt^4}{\delta}\right)}\right)^2\right) \\
&\leq \sum_{u=C_1 u_i^*(t)+1}^{\infty} \exp\left(-2u\Delta_i^2\left(\frac{f_i - c}{\Delta_i} - \frac{1}{u^{\frac{\gamma}{2}}}\sqrt{\frac{1}{2\Delta_i^2}\ln\left(\frac{nkt^4}{\delta}\right)}\right)^2\right) \\
&\leq \sum_{u=C_1 u_i^*(t)+1}^{\infty} \exp\left(-2u\Delta_i^2\left(\frac{f_i - c}{\Delta_i} - \frac{1}{2}\left(\frac{u_i^*(t)}{u}\right)^{\frac{\gamma}{2}}\right)^2\right). \qquad \text{(A.7)}
\end{aligned}$$

In this last expression, we have that $u > C_1 u_i^*(t)$, which implies, with $C_1 > 1$, that

$$\frac{1}{2}\left(\frac{u_i^*(t)}{u}\right)^{\frac{\gamma}{2}} < \frac{1}{2}\frac{1}{C_1^{\frac{\gamma}{2}}} < \frac{1}{2}.$$

At the same time, we have by definition of Δ_i that

$$\frac{1}{2} \leq \frac{f_i - c}{\Delta_i}.$$

Combining both inequalities, we get that

$$\begin{aligned}
&\left(\frac{f_i - c}{\Delta_i} - \frac{1}{2}\left(\frac{u_i^*(t)}{u}\right)^{\frac{\gamma}{2}}\right) \geq \left(\frac{1}{2} - \frac{1}{2}\frac{1}{C_1^{\frac{\gamma}{2}}}\right)^2 \\
\implies &\exp\left(-2u\Delta_i^2\left(\frac{f_i - c}{\Delta_i} - \frac{1}{2}\left(\frac{u_i^*(t)}{u}\right)^{\frac{\gamma}{2}}\right)\right) \leq \exp\left(-2u\Delta_i^2\left(\frac{1}{2} - \frac{1}{2}\frac{1}{C_1^{\frac{\gamma}{2}}}\right)^2\right)
\end{aligned}$$

We write $\tilde{C} \stackrel{\text{def}}{=} \left(\frac{1}{2} - \frac{1}{2}\frac{1}{C_1^{\frac{\gamma}{2}}}\right)^2$ and inject this result in Equation A.7:

$$\mathbb{P}\left(\left(u_i^{g,t} > C_1 u_i^*(t)\right) \wedge \textsc{Needy}_i^{g,t} \,\middle|\, \Delta_i > \frac{\epsilon}{2}\right) \leq \sum_{u=C_1 u_i^*(t)+1}^{\infty} \exp\left(-2\tilde{C}u\Delta_i^2\right).$$

Then, by remarking that the function $g : u \mapsto \exp\left(-2\tilde{C}u\Delta_i^2\right)$ is strictly decreasing, one can upper-bound the sum of $u \mapsto g(u)$ by the integral of $u \mapsto g(u-1)$, which implies

$$\begin{aligned}
\mathbb{P}\left(\left(u_i^{g,t} > C_1 u_i^*(t)\right) \wedge \textsc{Needy}_i^{g,t} \,\middle|\, \Delta_i > \frac{\epsilon}{2}\right) &\leq \int_{C_1 u_i^*(t)}^{\infty} \exp\left(-2\tilde{C}u\Delta_i^2\right) du \\
&\leq \frac{1}{2\tilde{C}\Delta_i^2} \exp\left(-2\Delta_i^2 C_1 \tilde{C} u_i^*(t)\right) \\
&\leq \frac{1}{2\tilde{C}\Delta_i^2} \exp\left(-2\Delta_i^2 C_1 \tilde{C}\left(\frac{2}{\Delta_i^2}\ln\left(\frac{nkt^4}{\delta}\right)\right)^{\frac{1}{\gamma}}\right) \\
&\leq \frac{1}{2\tilde{C}\Delta_i^2} \exp\left(-2\Delta_i^2 C_1 \tilde{C}\frac{2}{\Delta_i^2}\ln\left(\frac{nkt^4}{\delta}\right)\right) &\text{(A.8)} \\
&\leq \frac{1}{2\tilde{C}\Delta_i^2} \exp\left(-4C_1\tilde{C}\ln\left(\frac{nkt^4}{\delta}\right)\right) \\
&\leq \frac{1}{2\tilde{C}\Delta_i^2} \exp\left(-\ln\left(\frac{nkt^4}{\delta}\right)\right) &\text{(A.9)} \\
&\leq \frac{\delta}{2\tilde{C}\Delta_i^2 nkt^4}.
\end{aligned}$$

Equation A.8 comes from the fact that $\gamma < 1$, which implies $x^{\frac{1}{\gamma}} \geq x$ for any $x > 1$. To demonstrate Equation A.9, one can show that $4C_1\tilde{C} > 1$ by using the fact that we set $C_1 > 3^{\frac{2}{\gamma}}$.

Finally, to prove the result, we use the union bound over all the individuals:

$$
\begin{aligned}
\mathbb{P}\left(\exists i \in \textsc{Ind}^g, \left(u_i^{g,t} > C_1 u_i^*(t)\right) \wedge \textsc{Needy}_i^{g,t}\right) &\leq \sum_{i \in \textsc{Ind}^g} \mathbb{P}\left(\left(u_i^{g,t} > C_1 u_i^*(t)\right) \wedge \textsc{Needy}_i^{g,t}\right) \\
&\leq \sum_{i \in \textsc{Ind}^g} \mathbb{P}\left(\left(u_i^{g,t} > C_1 u_i^*(t)\right) \wedge \textsc{Needy}_i^{g,t} \;\middle|\; \Delta_i \leq \frac{\epsilon}{2}\right) + \\
&\qquad \mathbb{P}\left(\left(u_i^{g,t} > C_1 u_i^*(t)\right) \wedge \textsc{Needy}_i^{g,t} \;\middle|\; \Delta_i > \frac{\epsilon}{2}\right) \\
&\leq 0 + \sum_{\substack{i \in \textsc{Ind}^g \\ \Delta_i > \frac{\epsilon}{2}}} \frac{\delta}{2\tilde{C}\Delta_i^2 nkt^4} \\
&\leq \frac{\delta}{2\tilde{C}nkt^4} \sum_{i=1}^{n} \frac{1}{\Delta_i^2} \\
&\leq \frac{\delta H^{g,\frac{\epsilon}{2}}}{2\tilde{C}nkt^4},
\end{aligned}
$$

which concludes the proof by defining the constant $C_2 \overset{\text{def}}{=} \frac{1}{2\tilde{C}} > 0$. □

We introduce a third lemma, borrowed from [2] (Lemma 2), showing that if the algorithm does not terminate and $\textsc{Cross}^{g,t}$ is not verified, then $h_*^{g,t}$ or $l_*^{g,t}$ is necessarily needy. As the assumptions are the same as [2], we refer the reader to this paper for a formal proof. This result suggests that both individuals $h_*^{g,t}$ and $l_*^{g,t}$ are good candidates for sampling, in order to reach the termination criterion of Equation 2.

Lemma 3.4. *(Lemma 2 of [2]) At any step t of generation g, we have that*

$$
\neg \textsc{Cross}^{g,t} \wedge \neg \textsc{Term}^{g,t} \implies \textsc{Needy}_{h_*^{g,t}}^{g,t} \vee \textsc{Needy}_{l_*^{g,t}}^{g,t}.
$$

The following lemma gives a lower-bound on the number of steps t after which $2 + \sum_{i \in \textsc{Ind}^g} 32 u_i^*(t) \leq t$. It will be used in Lemma 3.6 to indicate the number of steps required for termination with a bounded probability.

Lemma 3.5. *At generation g, there exists a constant $C_3 > 0$ such that,*

$$
t \geq C_3 \left(H^{g,\frac{\epsilon}{2}} \ln\left(\frac{H^{g,\frac{\epsilon}{2}}}{\delta}\right)\right)^{\frac{1}{\gamma}} \implies 2 + 2 \sum_{i \in \textsc{Ind}^g} C_1 u_i^*(t) < t,
$$

with $C_1 > 3^{\frac{2}{\gamma}}$ defined in Lemma 3.3.

Proof. We develop the expression $2 + 2\sum_{i\in\textsc{Ind}^g} C_1 u_i^*(t)$ using the definition of $u_i^*(t)$ and derive an upper-bound:

$$\begin{aligned}
2 + 2\sum_{i\in\textsc{Ind}^g} C_1 u_i^*(t) &= 2 + 2\sum_{i\in\textsc{Ind}^g} C_1 \left\lceil \left(\frac{2}{\left[\Delta_i \wedge \frac{\epsilon}{2}\right]^2} \ln\left(\frac{nkt^4}{\delta}\right)\right)^{\frac{1}{\gamma}} \right\rceil && \text{(from Lemma 3.2)}\\
&\leq 2 + 2C_1 n + 2C_1 \sum_{i\in\textsc{Ind}^g} \left(\frac{2}{\left[\Delta_i \wedge \frac{\epsilon}{2}\right]^2} \ln\left(\frac{nkt^4}{\delta}\right)\right)^{\frac{1}{\gamma}} && \text{(as } \lceil x \rceil \leq 1 + x\text{)}\\
&\leq 2 + 2C_1 n + 2^{1+\frac{1}{\gamma}} C_1 \left(\ln\left(\frac{nkt^4}{\delta}\right)\right)^{\frac{1}{\gamma}} \sum_{i\in\textsc{Ind}^g} \left(\frac{1}{\left[\Delta_i \wedge \frac{\epsilon}{2}\right]^2}\right)^{\frac{1}{\gamma}}\\
&\leq 2 + 2C_1 n + 2^{1+\frac{1}{\gamma}} C_1 \left(\ln(k) + \ln\left(\frac{n}{\delta}\right) + 4\ln(t)\right)^{\frac{1}{\gamma}} \left(H^{g,\frac{\epsilon}{2}}\right)^{\frac{1}{\gamma}},
\end{aligned}$$

where we used the fact that $\sum_{i\in\textsc{Ind}^g} \left(\frac{1}{\left[\Delta_i \wedge \frac{\epsilon}{2}\right]^2}\right)^{\frac{1}{\gamma}} \leq \left(\sum_{i\in\textsc{Ind}^g} \frac{1}{\left[\Delta_i \wedge \frac{\epsilon}{2}\right]^2}\right)^{\frac{1}{\gamma}} = \left(H^{g,\frac{\epsilon}{2}}\right)^{\frac{1}{\gamma}}$ as $\frac{1}{\gamma} > 1$ and for all $i \in \textsc{Ind}^g$, $\frac{1}{\left[\Delta_i \wedge \frac{\epsilon}{2}\right]^2} \geq 1$. To simplify the notation, let us write $\varphi(t)$ the right-hand side of the last inequality:

$$\varphi(t) \stackrel{\text{def}}{=} 2 + 2C_1 n + 2^{1+\frac{1}{\gamma}} C_1 \left(\ln(k) + \ln\left(\frac{n}{\delta}\right) + 4\ln(t)\right)^{\frac{1}{\gamma}} \left(H^{g,\frac{\epsilon}{2}}\right)^{\frac{1}{\gamma}},$$

and let $T = C_3 \left(H^{g,\frac{\epsilon}{2}} \ln\left(\frac{H^{g,\frac{\epsilon}{2}}}{\delta}\right)\right)^{\frac{1}{\gamma}}$, $C_3 > 0$. We now prove that there indeed exists $C_3 > 0$ such that $\varphi(T) \leq T$. The proof will then follow easily by remarking that $t \mapsto \varphi(t)$ is a polylogarithmic function and is thus dominated by the identity function after reaching a certain constant value of t. Replacing T, we have:

$$\begin{aligned}
\varphi(T) &= 2 + 2C_1 n + 2^{1+\frac{1}{\gamma}} C_1 \left(H^{g,\frac{\epsilon}{2}}\right)^{\frac{1}{\gamma}} \left(\ln(k) + \ln\left(\frac{n}{\delta}\right) + 4\ln(C_3) + \frac{4}{\gamma}\ln\left(H^{g,\frac{\epsilon}{2}}\right) + \frac{4}{\gamma}\ln\left(\ln\left(\frac{H^{g,\frac{\epsilon}{2}}}{\delta}\right)\right)\right)^{\frac{1}{\gamma}}\\
&\leq 2 + 2C_1 n + 2^{1+\frac{1}{\gamma}} C_1 \left(H^{g,\frac{\epsilon}{2}}\right)^{\frac{1}{\gamma}} \left(\ln(k) + \ln\left(\frac{n}{\delta}\right) + 4\ln(C_3) + \frac{4}{\gamma}\ln\left(H^{g,\frac{\epsilon}{2}}\right) + \frac{4}{\gamma}\ln\left(\frac{H^{g,\frac{\epsilon}{2}}}{\delta}\right)\right)^{\frac{1}{\gamma}}\\
&\leq 2^{1+\frac{1}{\gamma}} C_1 \left(H^{g,\frac{\epsilon}{2}}\right)^{\frac{1}{\gamma}} \left(3 + \ln(k) + \ln\left(\frac{n}{\delta}\right) + 4\ln(C_3) + \frac{4}{\gamma}\ln\left(H^{g,\frac{\epsilon}{2}}\right) + \frac{4}{\gamma}\ln\left(\frac{H^{g,\frac{\epsilon}{2}}}{\delta}\right)\right)^{\frac{1}{\gamma}}, && \text{(A.10)}
\end{aligned}$$

Where we used the three facts that $n \leq H^{g,\frac{\epsilon}{2}} \leq \left(H^{g,\frac{\epsilon}{2}}\right)^{\frac{1}{\gamma}}$, $2^{1+\frac{1}{\gamma}} C_1 H^{g,\frac{\epsilon}{2}} > 1$ and $x + y^{\frac{1}{\gamma}} < (x+y)^{\frac{1}{\gamma}}$ for any $x, y \geq 1$. Recall that

$$2 \leq n \leq H^{g,\frac{\epsilon}{2}} \leq \frac{H^{g,\frac{\epsilon}{2}}}{\delta}.$$

We use this fact to upper bound $\ln\left(\frac{n}{\delta}\right)$ and $\ln\left(H^{g,\frac{\epsilon}{2}}\right)$ by $\ln\left(\frac{H^{g,\frac{\epsilon}{2}}}{\delta}\right)$ in Equation A.10. We can in turn factorize everything by $\ln\left(\frac{H^{g,\frac{\epsilon}{2}}}{\delta}\right)$, yielding

$$\varphi(T) \leq 2^{1+\frac{1}{\gamma}} C_1 \left(H^{g,\frac{\epsilon}{2}}\right)^{\frac{1}{\gamma}} \left(3 + \ln(k) + 1 + 4\ln(C_3) + \frac{4}{\gamma} + \frac{4}{\gamma}\right)^{\frac{1}{\gamma}} \left(\ln\left(\frac{H^{g,\frac{\epsilon}{2}}}{\delta}\right)\right)^{\frac{1}{\gamma}}$$
$$\leq 2^{1+\frac{1}{\gamma}} C_1 \left(4 + \ln(k) + 4\ln(C_3) + \frac{8}{\gamma}\right)^{\frac{1}{\gamma}} \left(H^{g,\frac{\epsilon}{2}} \ln\left(\frac{H^{g,\frac{\epsilon}{2}}}{\delta}\right)\right)^{\frac{1}{\gamma}}$$

We chose C_3 such that $C_3 \geq 2^{1+\frac{1}{\gamma}} C_1 \left(4 + \ln(k) + 4\ln(C_3) + \frac{8}{\gamma}\right)^{\frac{1}{\gamma}}$, which is always proven to exist as all polylogarithmic functions are dominated by any polynomial, particularly the identity function. For such a choice of C_3, we thus have that

$$\varphi(T) \leq C_3 \left(H^{g,\frac{\epsilon}{2}} \ln\left(\frac{H^{g,\frac{\epsilon}{2}}}{\delta}\right)\right)^{\frac{1}{\gamma}} = T$$

Now, by remarking that $t \mapsto \varphi(t)$ is a polylogarithmic function, specifically written in the form

$$\varphi(t) = A \ln(Bt)^{\frac{1}{\gamma}} + C,$$

where A, B, and C are positive constants, we have that it is $o\left((Bt)^{\tilde{\epsilon}}\right)$ for any exponent $\tilde{\epsilon} > 0$. Thus, for any positive constant $\bar{\epsilon} > 0$, there exists a constant t_0 such that $\varphi(t) \leq \bar{\epsilon}(Bt)^{\tilde{\epsilon}}$ for any $t \geq t_0$. By picking adequately small values for having $\bar{\epsilon}(Bt)^{\tilde{\epsilon}} \leq t$, we thus have the guarantee that there exists a constant t_0 such that $\varphi(t) < t$ for any $t \geq t_0$. We use this fact to chose C_3 large enough for T to be larger than t_0. We thus have that $2 + 2\sum_{i \in \textsc{Ind}^g} C_1 u_i^*(t) < t$ for any $t \geq T$, which completes the proof. □

We now consider the probability of non termination after t steps during generation g, and show that after a certain threshold on t, this probability is bounded by a decreasing value with t.

Lemma 3.6. *During generation g, for any $t \geq 2C_3 \left(H^{g,\frac{\epsilon}{2}} \ln\left(\frac{H^{g,\frac{\epsilon}{2}}}{\delta}\right)\right)^{\frac{1}{\gamma}}$ (where C_3 is defined in Lemma 3.5), the probability that LUCIE has not terminated after t steps is at most $\frac{C_4\delta}{t^2}$, with C_4 a strictly positive constant.*

Proof. Consider $T = C_3 \left(H^{g,\frac{\epsilon}{2}} \ln\left(\frac{H^{g,\frac{\epsilon}{2}}}{\delta}\right)\right)^{\frac{1}{\gamma}}$, and E_1, E_2 the events described in Lemma 3.1 and 3.3 for $t \in \{T, \ldots, 2T-1\}$, defined as

$$E_1 \overset{\text{def}}{=} \exists t \in \{T, \ldots, 2T-1\}, \textsc{Cross}^{g,t},$$
$$E_2 \overset{\text{def}}{=} \exists t \in \{T, \ldots, 2T-1\}, \exists i \in \textsc{Ind}^g, \left(u_i^{g,t} > C_1 u_i^*(t)\right) \wedge \textsc{Needy}_i^{g,t}.$$

We first demonstrate the following implication:

$$\neg E_1 \wedge \neg E_2 \implies \exists t \leq 2T-1, \textsc{Term}^{g,t} \tag{A.11}$$

Suppose for now that $\neg E_1 \wedge \neg E_2$. Let $N_{\text{non-term}}$ be the random variable of the number of steps during which $\neg\textsc{Term}^{g,t}$ is true, for $t \in \{1, \ldots, 2T-1\}$. Our goal is to show that, necessarily, $N_{\text{non-term}} < 2T-1$. This would imply that there exists $t \leq 2T-1$ such that $\textsc{Term}^{g,t}$. Recall that if $\textsc{Term}^{g,t}$ is true, then for all $t' \geq t$, $\textsc{Term}^{g,t'}$ is also true.

We distinguish two cases. If $\textsc{Term}^{g,T}$ is true, then we have $N_{\text{non-term}} \leq T < 2T - 1$. Else, assume the stopping criterion has not been reached at step T, *i.e.*, $\neg\textsc{Term}^{g,T}$. Let N_{remain} be the random variable of the number of steps during which $\neg\textsc{Term}^{g,t}$ for $t \in \{T, \ldots, 2T-1\}$, defined as:

$$N_{\text{remain}} = \sum_{t=T}^{2T-1} \mathbf{1}\left(\neg\textsc{Term}^{g,t}\right)$$

Since $\neg E_1$ is true, then for all $t \in \{T, \ldots, 2T-1\}$, $\neg\textsc{Cross}^{g,t}$ is true. Consequently, we can write N_{remain} as

$$N_{\text{remain}} = \sum_{t=T}^{2T-1} \mathbf{1}\left(\neg\textsc{Term}^{g,t} \wedge \neg\textsc{Cross}^{g,t}\right).$$

By Lemma 3.4, we have that $\neg\textsc{Term}^{g,t} \wedge \neg\textsc{Cross}^{g,t} \implies \textsc{Needy}^{g,t}_{h^{g,t}_*} \vee \textsc{Needy}^{g,t}_{l^{g,t}_*}$ for any $t \in \mathbb{N}$. Hence we can upper-bound N_{remain} by the number of times the sampling candidates are needy:

$$\begin{aligned} N_{\text{remain}} &\leq \sum_{t=T}^{2T-1} \mathbf{1}\left(\textsc{Needy}^{g,t}_{h^{g,t}_*} \vee \textsc{Needy}^{g,t}_{l^{g,t}_*}\right) \\ &\leq \sum_{t=T}^{2T-1} \sum_{i \in \textsc{Ind}^g} \mathbf{1}\left(\left(i = h^{g,t}_* \vee i = l^{g,t}_*\right) \wedge \textsc{Needy}^{g,t}_i\right). \end{aligned} \tag{A.12}$$

Since $\neg E_2$ is true, then for any individual $i \in \textsc{Ind}^g$, either the event $\neg\textsc{Needy}^{g,t}_i$ is true, either $\left(u^{g,t}_i \leq C_1 u^*_i(t)\right)$:

$$\begin{aligned} \neg E_2 &\iff \forall t \in \{T, \ldots, 2T-1\}, \forall i \in \textsc{Ind}^g, \left(u^{g,t}_i \leq C_1 u^*_i(t)\right) \vee \neg\textsc{Needy}^{g,t}_i \\ &\iff \forall t \in \{T, \ldots, 2T-1\}, \forall i \in \textsc{Ind}^g, \textsc{Needy}^{g,t}_i \implies u^{g,t}_i \leq C_1 u^*_i(t). \end{aligned}$$

Using this along with the fact that $t \mapsto u^*_i(t)$ is an increasing function, we have that

$$\begin{aligned} N_{\text{remain}} &\leq \sum_{t=T}^{2T-1} \sum_{i \in \textsc{Ind}^g} \mathbf{1}\left(\left(i = h^{g,t}_* \vee i = l^{g,t}_*\right) \wedge \left(u^{g,t}_i \leq C_1 u^*_i(t)\right)\right) && (\neg E_2) \\ &\leq \sum_{t=T}^{2T-1} \sum_{i \in \textsc{Ind}^g} \mathbf{1}\left(\left(i = h^{g,t}_* \vee i = l^{g,t}_*\right) \wedge \left(u^{g,t}_i \leq C_1 u^*_i(2T)\right)\right) && (t < 2T) \\ &\leq \sum_{i \in \textsc{Ind}^g} \sum_{t=T}^{2T-1} \mathbf{1}\left(\left(i = h^{g,t}_* \vee i = l^{g,t}_*\right) \wedge \left(u^{g,t}_i \leq C_1 u^*_i(2T)\right)\right) \\ &\leq \sum_{i \in \textsc{Ind}^g} C_1 u^*_i(2T). \end{aligned}$$

The last step of this derivation comes from the fact that, at step t, the number of times an individual is selected (*i.e.*, $i = h^{g,t}_* \vee i = l^{g,t}_*$) *and* its number of samples is lesser than $C_1 u^*_i(2T)$ cannot exceed $C_1 u^*_i(2T)$. Indeed, each time the individual is sampled, its number of samples is increased by 1, which, along with the event $u^{g,t}_i \leq C_1 u^*_i(2T)$ cannot happen more than $C_1 u^*_i(2T)$ times. As $T = C_3 \left(H^{g,\frac{\epsilon}{2}} \ln\left(\frac{H^{g,\frac{\epsilon}{2}}}{\delta}\right)\right)^{\frac{1}{\gamma}}$, according to Lemma 3.5, we thus have that

$$N_{\text{remain}} \leq \sum_{i \in \textsc{Ind}^g} C_1 u^*_i(2T) < \frac{2T-2}{2} = T - 1.$$

Overall, we have in this second case that

$$N_{\text{non-term}} = T + N_{\text{remain}} < 2T - 1.$$

Thus, in any case, $N_{\text{non-term}} < 2T - 1$, which concludes our proof of the implication described in Equation A.11. Its counterpart is the following:

$$\forall t \leq 2T - 1, \neg \textsc{Term}^{g,t} \implies E_1 \vee E_2$$

Hence, the probability of verifying $\neg\textsc{Term}^{g,t}$ for the first time after $2T$ steps is upper-bounded by the probability of $E_1 \vee E_2$, for which we can use Lemma 3.1 and Lemma 3.3 to find an upper-bound:

$$\begin{aligned}
\mathbb{P}(E_1 \vee E_2) &\leq \mathbb{P}(E_1) + \mathbb{P}(E_2) \\
&\leq \mathbb{P}\left(\exists t \in \{T, \dots, 2T-1\}, \textsc{Cross}^{g,t}\right) \\
&\quad + \mathbb{P}\left(\exists t \in \{T, \dots, 2T-1\}, \exists i \in \textsc{Ind}^{g}, \left(u_i^{g,t} > C_1 u_i^*(t)\right) \wedge \textsc{Needy}_i^{g,t}\right) \\
&\leq \sum_{t=T}^{2T-1} \mathbb{P}\left(\textsc{Cross}^{g,t}\right) + \sum_{t=T}^{2T-1} \mathbb{P}\left(\exists i \in \textsc{Ind}^{g}, \left(u_i^{g,t} > C_1 u_i^*(t)\right) \wedge \textsc{Needy}_i^{g,t}\right) && \text{(union bound)} \\
&\leq \sum_{t=T}^{2T-1} \frac{\delta\zeta(2)}{kt^4} + C_2 \frac{\delta H^{g,\frac{\epsilon}{2}}}{nkt^4} && \text{(from Lemma 3.1 and Lemma 3.3)} \\
&\leq \sum_{t=T}^{2T-1} \frac{\delta}{kt^4}\left(\zeta(2) + C_2 \frac{H^{g,\frac{\epsilon}{2}}}{n}\right) \\
&\leq \sum_{t=T}^{2T-1} \frac{\delta}{kT^4}\left(\zeta(2) + C_2 \frac{H^{g,\frac{\epsilon}{2}}}{n}\right) \\
&\leq \frac{\delta}{kT^2}\left(\frac{\zeta(2)}{T} + \frac{C_2}{n}\frac{H^{g,\frac{\epsilon}{2}}}{T}\right) \\
&\leq \frac{C_4\delta}{T^2},
\end{aligned}$$

where C_4 is a positive constant. The existence of C_4 comes from the fact that all quantities are upper-bounded by positive constants. Namely, as $n \geq 2$ and $\delta \leq 0.5$, if we write $T_{\min} \stackrel{\text{def}}{=} C_3 \left(2 \ln\left(\frac{2}{0.5}\right)\right)^{\frac{1}{\gamma}}$ the minimum value of T, we have that

$$\frac{\zeta(2)}{T} \leq \frac{\zeta(2)}{T_{\min}}, \quad \frac{C_2}{n} \leq \frac{C_2}{2}, \text{ and, } \quad \frac{H^{g,\frac{\epsilon}{2}}}{T} = \frac{1}{C_3\left(H^{g,\frac{\epsilon}{2}}\right)^{\frac{1}{\gamma}-1}\left(\ln\left(\frac{H^{g,\frac{\epsilon}{2}}}{\delta}\right)\right)^{\frac{1}{\gamma}}} \leq \frac{1}{C_3\,(2)^{\frac{1}{\gamma}-1}\left(\ln\left(\frac{2}{0.5}\right)\right)^{\frac{1}{\gamma}}}.$$

The proof is concluded by remarking that this is true for any $T \geq 2C_3\left(H^{g,\frac{\epsilon}{2}} \ln\left(\frac{H^{g,\frac{\epsilon}{2}}}{\delta}\right)\right)^{\frac{1}{\gamma}}$. □

Finally, Lemma 3.6 allows to demonstrate Theorem 4.2 as follows.

Proof. Following Lemma 3.6, consider $T^* = C_3\left(H^{g,\frac{\epsilon}{2}} \ln\left(\frac{H^{g,\frac{\epsilon}{2}}}{\delta}\right)\right)^{\frac{1}{\gamma}}$. At generation g, the sample complexity, denoted by SC, is defined as the number of steps where the binary random variable $\neg\textsc{Term}^{g,t}$ is true. As two individuals are sampled at each step, we have that $SC = 2\sum_{t=1}^{\infty} \neg\textsc{Term}^{g,t}$. The result follows from taking the expectation and applying Lemma 3.6:

$$\begin{aligned}
\mathbb{E}\left(SC\right) &= 2\sum_{t=1}^{\infty}\mathbb{E}\left(\neg\textsc{Term}^{g,t}\right)\\
&= 2\sum_{t=1}^{\infty}0\times\mathbb{P}\left(\textsc{Term}^{g,t}\right)+1\times\mathbb{P}\left(\neg\textsc{Term}^{g,t}\right)\\
&= 2\sum_{t=1}^{T^*}\mathbb{P}\left(\neg\textsc{Term}^{g,t}\right)+2\sum_{t=T^*+1}^{\infty}\mathbb{P}\left(\neg\textsc{Term}^{g,t}\right)\\
&\leq 2T^*+2\sum_{t=T^*+1}^{\infty}\frac{C_4\delta}{T^2}\\
&\leq 2C_3\left(H^{g,\frac{\epsilon}{2}}\ln\left(\frac{H^{g,\frac{\epsilon}{2}}}{\delta}\right)\right)^{\frac{1}{\gamma}}+C_4\delta\zeta(2).
\end{aligned}$$

□

4 PROOF OF THEOREM 4.3

Claim: For any noise model verifying H1, setting $\epsilon < 1$ and $\delta \leq kc/N$, (1+1) LUCIE optimizes the stochastic OneMax problem in $O\left(N\ln\left(N\right)\right)$ number of generations.

Proof. To prove the result, we show that applying (1+1) LUCIE amounts to the same setting as Theorem 4 of [1] while transferring the assumption on the degree of stochasticity (Equation 1 in their paper) to the parameter δ. Specifically, they prove that (1+1) Evolutionary Algorithm converges in $O\left(N\ln\left(N\right)\right)$ generations under a restrictive assumptions on the problem's degree of stochasticity that we detail now along with their notations.

Given an individual $i \in \textsc{Ind}^g$, [1] introduce the notion of *observation* of the individual's fitness as a random variable, written X_l, for $f_i = l$ and $l \in \{0,\ldots,N\}$. Recall that, in this setting, only the observed fitness could be accessed by an algorithm as the evaluation protocol is subject to noise. To ease the comparison between (1+1) LUCIE and (1+1) EA, we will write $O\left(i\right)$ the observation of the fitness of individual i. At elite selection phase of generation g, the observation of the individual's fitness is different between both algorithms and we have:

$$\begin{aligned}
&O\left(i\right) \sim X_{f_i} \text{ for EA,}\\
&O\left(i\right) = \hat{f}_i^{g,t} \text{ for LUCIE, given that } \textsc{Term}^{g,t} \text{ is true.}
\end{aligned}$$

In other words, during the selection of elites, we observe a single realization of the individual's fitness random variable in (1+1) EA while we observe the empirical mean in (1+1) LUCIE. Alongside, there is one assumption, restricting the level of noise that the algorithm can handle, made in Theorem 4 of [1]:

H2: $\forall l < N, \forall c \in]0, \frac{1}{9}[$,

$$\mathbb{P}\left(X_l < X_{l+1}\right) \geq 1 - c\frac{N-l}{N}.$$

Intuitively, successful elite selection for two individuals of fitness l and $l+1$ must happen with a probability that is *not too low* for (1+1) EA to converge. This effect is accentuated with individuals having a true fitness close to N. Using our notations, this amounts to write:

H2: $\forall l < N, \forall c \in]0, \frac{1}{9}[, \forall i_1, i_2 \in \textsc{Ind}$,

$$\mathbb{P}\left(O\left(i_1\right) < O\left(i_2\right) \,\middle|\, f_{i_1} = l, f_{i_2} = l+1\right) \geq 1 - c\frac{N-l}{N}.$$

Let us now verify that H2 is verified in the case of LUCIE. Consider two individuals $i_1, i_2 \in \textsc{Ind}$ and $l < N$.

$$\begin{aligned}\mathbb{P}\left(O\left(i_1\right) < O\left(i_2\right) \,\middle|\, f_{i_1} = l, f_{i_2} = l+1\right) &= \mathbb{P}\left(\hat{f}_{i_1}^{g,t} < \hat{f}_{i_2}^{g,t} \,\middle|\, f_{i_1} = l, f_{i_2} = l+1\right) \\ &= 1 - \mathbb{P}\left(\hat{f}_{i_1}^{g,t} \geq \hat{f}_{i_2}^{g,t} \,\middle|\, f_{i_1} = l, f_{i_2} = l+1\right) \end{aligned} \tag{A.13}$$

The last term of the right-hand-side is linked with the failure probability of Theorem 4.1. Assume that $f_{i_1} = l$ and $f_{i_2} = l+1$, we show that the event of $\hat{f}_{i_1}^{g,t} \geq \hat{f}_{i_2}^{g,t}$ implies that either i_1 or i_2 is not well-behaved (Equation A.4). Assume that i_1 is well-behaved. As $i_1 \in \textsc{Bot}^g$, we have that

$$\hat{f}_{i_1}^{g,t} \leq f_{i_1} + \beta\left(u_{i_1}^{g,t}, t\right).$$

Necessarily, as fitness are *observed*, this means in LUCIE that the stopping criterion (Equation 2) has been reached, here by suggesting i_1 as elite individual:

$$\hat{f}_{i_1}^{g,t} - \beta\left(u_{i_1}^{g,t}, t\right) + \epsilon \geq \hat{f}_{i_2}^{g,t} + \beta\left(u_{i_2}^{g,t}, t\right).$$

We now set $\epsilon < 1$, so that we verify

$$f_{i_1} < f_{i_2} - \epsilon$$

Combining those inequalities, we get that

$$f_{i_2} > f_{i_1} + \epsilon > \hat{f}_{i_1}^{g,t} - \beta\left(u_{i_1}^{g,t}, t\right) + \epsilon > \hat{f}_{i_2}^{g,t} + \beta\left(u_{i_2}^{g,t}, t\right),$$

which implies that i_2 is not well-behaved. Therefore, the event of $\hat{f}_{i_1}^{g,t} \geq \hat{f}_{i_2}^{g,t}$ implies that either i_1 or i_2 is not well-behaved. Hence, we have that

$$\begin{aligned}\mathbb{P}\left(\hat{f}_{i_1}^{g,t} \geq \hat{f}_{i_2}^{g,t} \,\middle|\, f_{i_1} = l, f_{i_2} = l+1\right) &\leq \mathbb{P}\left(\neg WB_{i_1}^{g,t}(u_{i_1}^{g,t}) \vee \neg WB_{i_2}^{g,t}(u_{i_2}^{g,t}) \,\middle|\, f_{i_1} = l, f_{i_2} = l+1\right) \\ &\leq \mathbb{P}\left(\exists i \in \textsc{Ind}^g, \neg WB_{i}^{g,t}(u_{i}^{g,t}) \,\middle|\, f_{i_1} = l, f_{i_2} = l+1\right) \\ &\leq \sum_{i=1}^{n} \mathbb{P}\left(\neg WB_{i}^{g,t}(u_{i}^{g,t}) \,\middle|\, f_{i_1} = l, f_{i_2} = l+1\right) \\ &\leq \sum_{i=1}^{n}\sum_{u=1}^{\infty} \mathbb{P}\left(\neg WB_{i}^{g,t}(u) \,\middle|\, f_{i_1} = l, f_{i_2} = l+1\right) \\ &\leq \sum_{i=1}^{n}\sum_{u=1}^{\infty} \exp\left(-2u\beta\left(u,t\right)^2\right) \qquad \text{(Hoeffding's bound)} \\ &\leq \sum_{i=1}^{n}\sum_{u=1}^{\infty} \frac{\delta}{nkt^4u^2} \\ &\leq \frac{\delta\zeta(2)}{kt^4} \\ &\leq \frac{\delta}{k},\end{aligned}$$

as $\zeta(2) < 2$. Injected in Equation A.13, we get that

$$\mathbb{P}\left(O\left(i_1\right) < O\left(i_2\right) \,\middle|\, f_{i_1} = l, f_{i_2} = l+1\right) \geq 1 - \frac{\delta}{k}.$$

To verify Assumption H2, we shall set $\delta/k \leq c(N-l)/N$, which is verified for all l for $\delta \leq kc/N$. Setting this condition on δ along with the condition $\epsilon < 1$ and $c < 1/9$ concludes the verification of the assumptions of Theorem 4 of [1]. In turn, applying this result concludes the proof. □

5 PROOF OF THEOREM 4.4

Claim: For any noise model verifying H3 and H4, setting $\epsilon < 1$ and $\delta \leq k/12N^2$, (1+1) LUCIE optimizes the stochastic LeadingOnes problem in $O\left(N^2\right)$ number of generations.

Proof. Similarly to the proof of Theorem 4.3, we demonstrate that applying (1+1) LUCIE amounts to the same setting as Theorem 11 of [1]. Again, the assumption on on the degree of stochasticity (Equation 3 in their paper) is transferred to a prerequisite on the value of δ. Borrowing the notations of [1], this assumption on the level of noise is the following:

H5: $\forall l \leq N, \forall c \in]0, \frac{1}{12}[$,

$$\mathbb{P}\left(X_l^{\text{opt}} < X_{l+1}^{\text{pes}}\right) \geq 1 - \frac{c}{lN}.$$

Borrowing the same notations as in the proof of Theorem 4.3, this amounts to the following:

H5: $\forall l \leq N, \forall c \in]0, \frac{1}{12}[, \forall i_1, i_2 \in \textsc{Ind}$,

$$\mathbb{P}\left(O\left(i_1\right) < O\left(i_2\right) \,\middle|\, i_1 = x_l^{\text{opt}}, i_2 = x_{l+1}^{\text{pes}}\right) \geq 1 - \frac{c}{lN},$$

with the fact that an observation in (1+1) EA is a single sample of the random variable, while it is the empirical mean of several samples in (1+1) LUCIE.

Let us now verify that H5 is verified in the case of LUCIE. Consider two individuals $i_1, i_2 \in \textsc{Ind}$ and $l < N$.

$$\begin{aligned}\mathbb{P}\left(O\left(i_1\right) < O\left(i_2\right) \,\middle|\, i_1 = x_l^{\text{opt}}, i_2 = x_{l+1}^{\text{pes}}\right) &= \mathbb{P}\left(\hat{f}_{i_1}^{g,t} < \hat{f}_{i_2}^{g,t} \,\middle|\, i_1 = x_l^{\text{opt}}, i_2 = x_{l+1}^{\text{pes}}\right) \\ &= 1 - \mathbb{P}\left(\hat{f}_{i_1}^{g,t} \geq \hat{f}_{i_2}^{g,t} \,\middle|\, i_1 = x_l^{\text{opt}}, i_2 = x_{l+1}^{\text{pes}}\right) \end{aligned} \tag{A.14}$$

The last term of the right-hand-side is linked with the failure probability of Theorem 4.1. Assume that $i_1 = x_l^{\text{opt}}$ and $i_2 = x_{l+1}^{\text{pes}}$. Hence, $f_{i_1} = l$ and $f_{i_2} = l+1$. From now, we will use the same arguments as in the proof of Theorem 4.1 where we proved that $\mathbb{P}\left(\hat{f}_{i_1}^{g,t} \geq \hat{f}_{i_2}^{g,t} \,\middle|\, f_{i_1} = l, f_{i_2} = l+1\right) \leq \frac{\delta\zeta(2)}{kt^4} \leq \frac{\delta}{k}$, yielding

$$\mathbb{P}\left(\hat{f}_{i_1}^{g,t} \geq \hat{f}_{i_2}^{g,t} \,\middle|\, i_1 = x_l^{\text{opt}}, i_2 = x_{l+1}^{\text{pes}}\right) \leq \frac{\delta}{k}.$$

Injecting in Equation (A.14), we get that

$$\mathbb{P}\left(O\left(i_1\right) < O\left(i_2\right) \,\middle|\, i_1 = x_l^{\text{opt}}, i_2 = x_{l+1}^{\text{pes}}\right) \geq 1 - \frac{\delta}{k}.$$

To verify H5, we shall set $\delta/k \leq c/lN$, which is verified for all l for $\delta \leq ck/N^2$. Setting this condition on δ along with the condition $\epsilon < 1$ and $c < 1/12$ concludes the verification of the assumptions of Theorem 11 of [1]. In turn, applying this result concludes the proof. □

6 BINARY EVOLUTION ADDITIONAL RESULTS ON ONEMAX AND LEADINGONES

Learning curves for binary evolution under posterior Gaussian noise for the OneMax and LeadingOnes tasks with vector size $N = 10$. Recall that noise samples are added to the true individual's fitness each time a fitness is sampled.

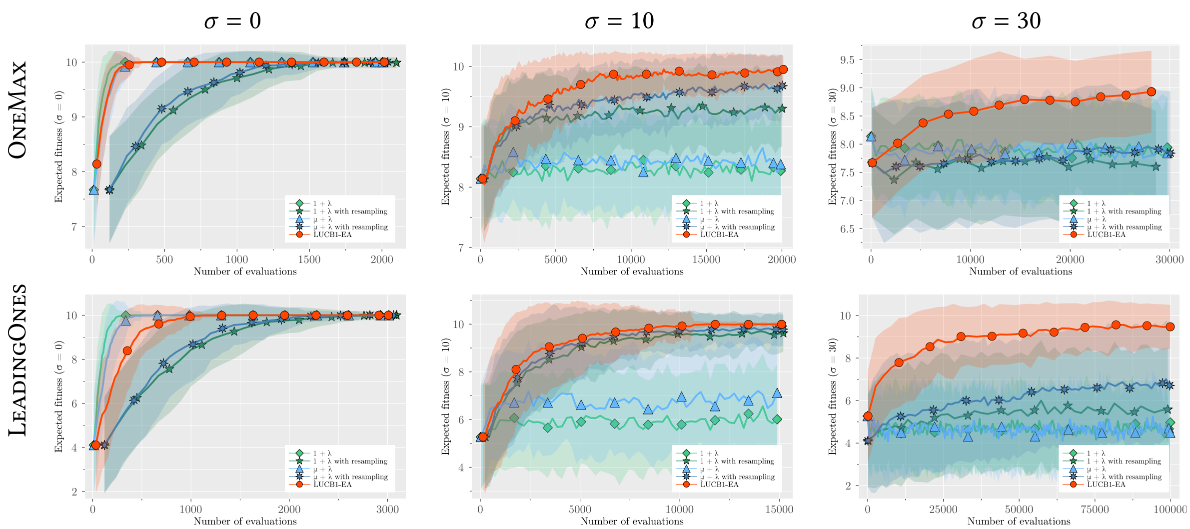

Table 1. Results for OneMax and LeadingOnes under posterior Gaussian noise.

7 NEUROEVOLUTION ADDITIONAL RESULTS ON ROBOTICS TASKS

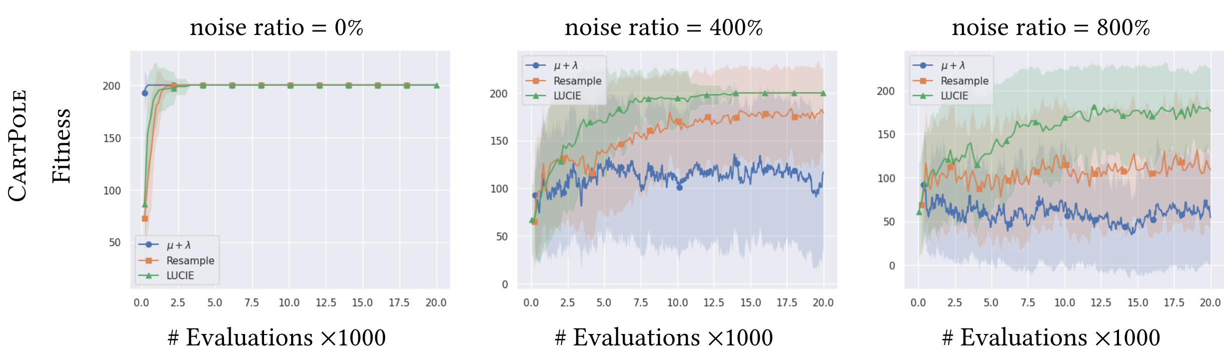

Table 2. Training fitness for CartPole neuroevolution under posterior uniform noise.

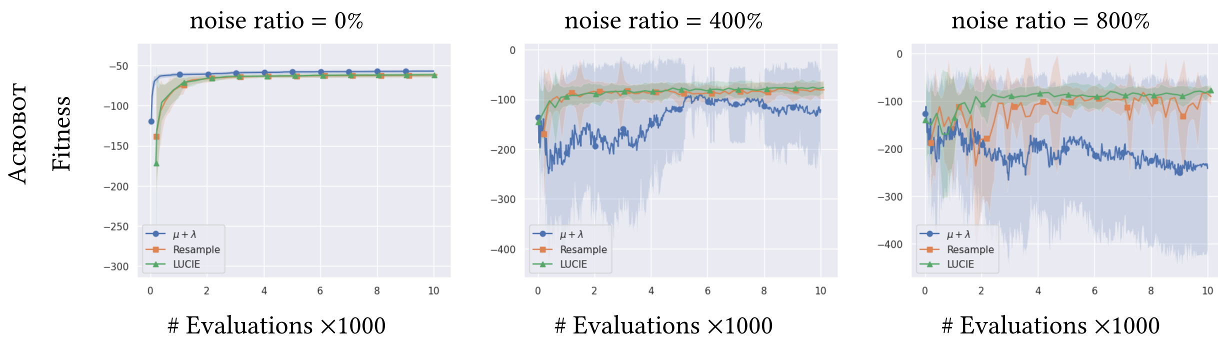

Table 3. Training fitness for Acrobot neuroevolution under posterior uniform noise.

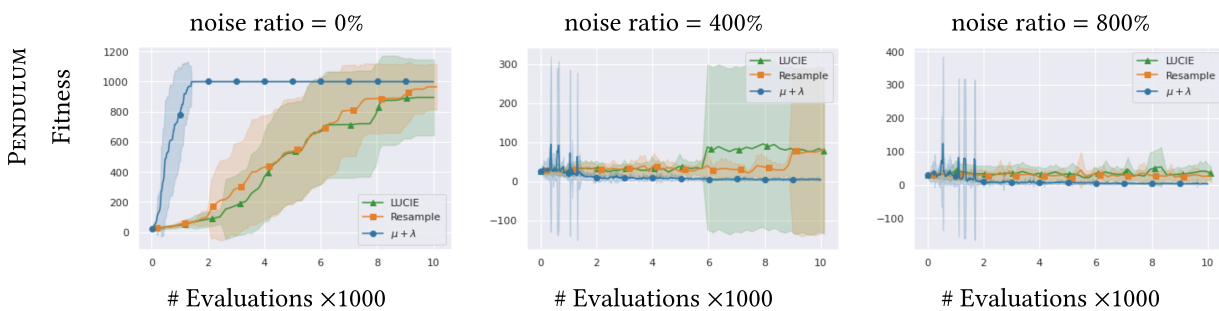

Table 4. Training fitness for Pendulum neuroevolution under posterior uniform noise.

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